

1. In lecture, we defined the *extended Hamming code* \mathcal{H}_8 of length $n = 8$ as the set of all binary 8-tuples $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ satisfying the following set of linear simultaneous equations (modulo 2):

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0 \tag{1}$$

$$x_0 + x_1 + x_2 + x_3 = 0 \tag{2}$$

$$x_0 + x_1 + x_4 + x_5 = 0 \tag{3}$$

$$x_0 + x_2 + x_4 + x_6 = 0. \tag{4}$$

We demonstrated that the extended Hamming code is at least 1-error correcting by exhibiting an explicit decoding rule (based on a syndrome table) capable of correcting all single bit errors.

- (a) List all code words of \mathcal{H}_8 .
 - (b) Determine a generator matrix for \mathcal{H}_8 .
 - (c) Determine a parity check matrix for \mathcal{H}_8 .
 - (d) Determine the dimension and rate of \mathcal{H}_8 .
2. Consider \mathcal{H}_8 once again.
- (a) Eliminate the first constraint, equation (1), and define a new code \mathcal{H}_8^* consisting of all 8-tuples satisfying constraints (2) through (4). What are the dimension and rate of \mathcal{H}_8^* ? Is \mathcal{H}_8^* still 1-error correcting? Explain mathematically.
 - (b) Eliminate the constraint (1) and also delete bit x_7 to produce a new code \mathcal{H}_7 consisting of binary 7-tuples. (This operation is referred to as *puncturing* the code \mathcal{H}_8 .) What are the dimension and rate of the new code? Is the code still 1-error correcting? Explain mathematically.
3. Consider the code \mathcal{H}_7 derived in the previous problem. Show that every possible received binary 7-tuple can be made into a code word by flipping at most one of its bits.
4. Let \mathcal{H}_7 be used on a *binary erasure channel* BEC(p), with erasure probability satisfying $0 < p < 1$. Show that the code is capable of correcting all error patterns with two or fewer erasures.
5. Consider the following 3×6 matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Which, if any, are generator matrices for a $[6, 3]$ linear binary code? Which, if any, are generator matrices for the same code?

6. Let a $r \times n$ binary matrix, with $r \leq n$, be chosen at random. What is the probability that it is the parity check matrix for an $[n, k = n - r]$ binary code?