1. In lecture, we defined the *extended Hamming code* $\mathcal{H}_8$ of length $n = 8$ as the set of all binary 8-tuples $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ satisfying the following set of linear simultaneous equations (modulo 2):

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 0 \quad (1)$$
$$x_0 + x_1 + x_2 + x_3 = 0 \quad (2)$$
$$x_0 + x_1 + x_4 + x_5 = 0 \quad (3)$$
$$x_0 + x_2 + x_4 + x_6 = 0. \quad (4)$$

We demonstrated that the extended Hamming code is at least 1-error correcting by exhibiting an explicit decoding rule (based on a syndrome table) capable of correcting all single bit errors.

(a) List all code words of $\mathcal{H}_8$.

(b) Determine a generator matrix for $\mathcal{H}_8$.

(c) Determine a parity check matrix for $\mathcal{H}_8$.

(d) Determine the dimension and rate of $\mathcal{H}_8$.

2. Consider the above definition of $\mathcal{H}_8$. Delete bit x_7 and eliminate the first constraint, equation (1), to produce the corresponding definition of a new code $\mathcal{H}_7$ consisting of binary 7-tuples. What are the dimension and rate of the new code? Is the code still 1-error correcting? Explain mathematically.

3. Consider the code $\mathcal{H}_7$ derived in the previous problem. Show that every possible received binary 7-tuple can be made into a code word by flipping at most one of its bits.

4. Consider a code defined by a bipartite graph having nodes $N_0, N_1, \ldots, N_6$ on the left representing constraints and nodes $x_0, x_1, \ldots, x_6$ on the right representing bits in the code word. Each constraint node N_i is connected to bit nodes x_i, x_{i+2}, x_{i+3} and x_{i+4}, where the subscripts are evaluated modulo 7. In particular, the constraint N_i corresponds to the linear homogeneous equation

$$x_i + x_{i+2} + x_{i+3} + x_{i+4} = 0.$$

(a) Find a generator matrix and parity check matrix for the code.

(b) What are the code parameters $[n, k, d_{\min}]$?

5. Consider the following 3×6 matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Which, if any, are generator matrices for a $[6, 3]$ linear binary code? Which, if any, are generator matrices for the same code?