References:

- Tekalp pages 411-415
- Papers by Eero Simoncelli and Edward Adelson
  - Orthogonal Pyramid Transforms for Image Coding
  - Chapter 4 from Subband Coding (edited by John Woods): Subband Transforms (This is available on internet):
    - http://www-bcs.mit.edu/people/adelson/publications/abstracts/simoncelli/subband.html

SUBBAND CODING

- Image is first filtered to create a set of images:
  - Each image contains a limited range of spatial frequencies
  - Images are called subbands.
  - Each subband has a reduced bandwidth compared to the original full-band image
    - The image may be downsampled
    - Filtering and subsampling is termed analysis stage
- Subbands are encoded using one or more coders

SUBBAND CODING

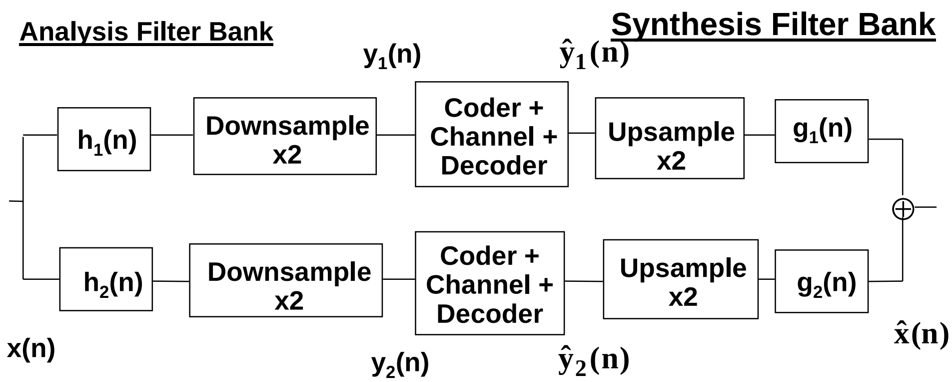
- Different bit rates or even different coding techniques can be used for each subband
  - Takes advantages of properties of the subband
  - Allowing for the coding errors to be distributed across the subbands in a visually optimal manner
- Reconstruction is achieved by upsampling the decoded subbands, applying appropriate filters, and adding the reconstructed subbands together
  - Called synthesis stage

IMAGE COMPRESSION AND PACKET VIDEO

SUBBAND CODING

- Formulation of subbands does not create any compression
  - Same number of samples is required to represent the subbands as is required for the original images.
- Subbands can be encoded more efficiently than the original image

1D, TWO-BAND SBC BLOCK DIAGRAM

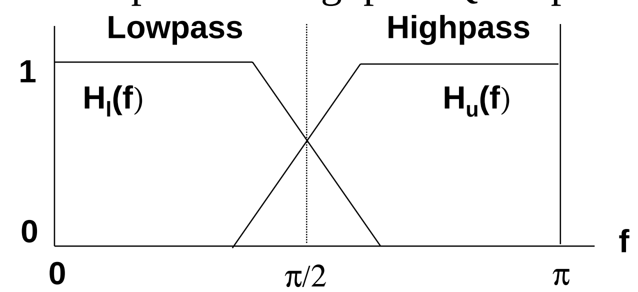


ANALYSIS/SYNTHESIS FILTERING

- Decompose a full-band 1D signal into two subbands ideally consist of a lowpass and a highpass filter set
  - frequency responses that are nonoverlapping, but contiguous
  - unity gain over their bandwidths
- Ideal filters are unrealizable in practice
  - Necessary to use filters with overlapping responses in order to prevent frequency gaps in signals represented by the subbands
  - Aliasing is introduced by overlapping filters when downsampled.

QUADRATURE MIRROR FILTERS

- QMF will allow alias-free reconstruction in the absence of coding errors.
- Filters exhibit mirror symmetry about π/2 radians
  - 1/4 of the normalized sampling frequency
- Idealized lowpass and highpass QMF pair:



QUADRATURE MIRROR FILTERS

- Philosophy of QMFs is to allow aliasing to be introduced by using overlapping filters for the analysis bank
- Design the synthesis filters in such a way that any aliasing is exactly cancelled out in the reconstruction process.
  - Filters are also designed so that overall amplitude and phase distortion are minimized or eliminated.
- Filters can be designed using z-transforms

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

OUTPUT SIGNALS FROM ANALYSIS BANK

- After downsampling by a factor of 2:

$$Y_l(f) = \frac{1}{2}\left[H_l(f/2)S(f/2) + H_l(-f/2)S(-f/2)\right]$$

$$Y_u(f) = \frac{1}{2}\left[H_u(f/2)S(f/2) + H_u(-f/2)S(-f/2)\right]$$

- Reconstructed signal is found by summing the outputs from the synthesis bank after upsampling:

$$\hat{S}(f) = G_l(f)\hat{Y}_l(2f) + G_u(f)\hat{Y}_u(2f)$$

OUTPUT SIGNALS FROM ANALYSIS BANK

- Ignoring any coding effects, set $\hat{Y}_i(f) = Y_i(f)$ for i=1,2, equation is equivalent to:

$$\hat{S}(f) = \frac{1}{2}\left[H_l(f)G_l(f) + H_u(f)G_u(f)\right]S(f) + \frac{1}{2}\left[H_l(-f)G_l(f) + H_u(-f)G_u(f)\right]S(-f)$$

OUTPUT SIGNALS FROM ANALYSIS BANK

- Second Term represents the aliased components (desired to be zero)
- Use the symmetric (linear phase) FIR Filters and let:

$$H_u(f) = H_l(-f) = H_l(f + 1/2)$$
$$G_l(f) = 2H_l(f)$$
$$G_u(f) = -2H_u(f)$$

- which are the requirements for QMF.

OUTPUT SIGNALS FROM ANALYSIS BANK

- The First Condition is equivalent to:

$$h_u(n) = (-1)^n h_l(n)$$

- the aliased term becomes zero and the reconstructed signal is:

$$\hat{S}(f) = \left[H_l^2(f) - H_u^2(f)\right]S(f)$$

- The overall transfer function for the system is:

$$T(f) = H_l^2(f) - H_u^2(f)$$

OUTPUT SIGNALS FROM ANALYSIS BANK

- Since both $H_l(f)$ and $H_u(f)$ have linear phase, the system introduces no phase distortion.
- Remaining problem is to eliminate amplitude distortion by making T(f) equal to 1 for every f.
  - This allows for perfect reconstruction:
- Only linear phase FIR capable of achieving this goal has exactly two taps
  - Frequency separation of the two-tap lowpass and highpass filters is quite poor

OUTPUT SIGNALS FROM ANALYSIS BANK

- Some Amplitude distortion must be tolerated.
- Filters can be designed using various optimization techniques employing constraints on:
  - passband ripple
  - stopband rejection
  - transition region bandwidth
- Larger filters with more taps, yield better coding performance at the expense of increased computational load.

OUTPUT SIGNALS FROM ANALYSIS BANK

- In addition, the filters must have an even number of taps:

$$T(e^{j\omega}) = \left[\left| H_l(e^{j\omega}) \right|^2 - (-1)^{(N-1)} \left| H_u(e^{j\omega}) \right|^2 \right] e^{-j\omega(N-1)}$$

- For odd N, the frequency response at $\omega=\pi/2$ is zero.

EXTENDING TO 2-D SIGNALS

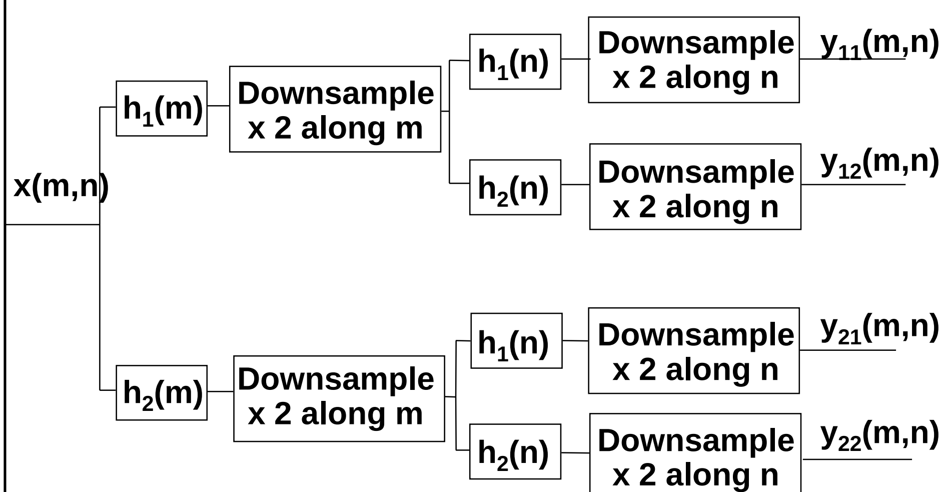
- 2D QMFs can be designed as separable filters
  - 1D filters can be applied first in one dimension, and then in the other dimension
  - Subsampling can be done after filtering in the first dimension to reduce the number of operations required for filtering in the other dimension.

EXTENDING TO 2-D SIGNALS

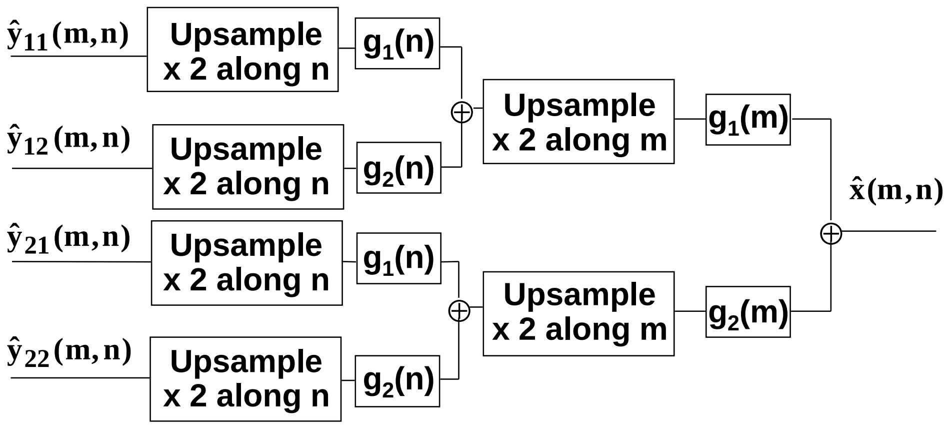
- Subbands are:
  - lowpass/lowpass
  - lowpass/highpass
  - highpass/lowpass
  - highpass/highpass
- If more than four subbands are desired, then the QMF structure can be applied repeatedly to one or more subbands

IMAGE COMPRESSION AND PACKET VIDEO

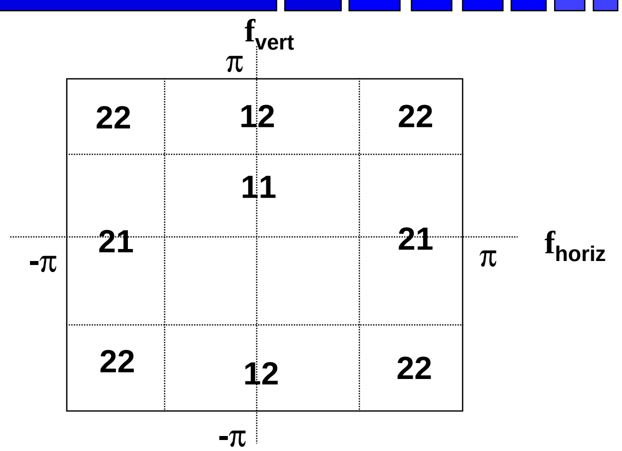
2-D, FOUR-BAND ANALYSIS BANK



2-D, FOUR-BAND SYNTHESIS BANK



APPROXIMATE FREQUENCY RANGES FOR A FOUR-BAND SPLIT



11- Lowpass/Lowpass
12 - Lowpass/Highpass
21- Highpass/Lowpass
22 - Highpass/Highpass

ERROR VARIANCE PRESERVATION

- Any error variance introduced in the subbands through quantization or deletion of subbands is preserved in the reconstructed signal
  - Sum of the subband variances is equal to the variance of the original image
- This is true for QMFs if and only if:

$$T(f) = H_l^2(f) - H_u^2(f) = 1$$

- Means of the lowest subbands are equal to the mean of the image
- Means of the remaining subbands are zero

CODING OF THE SUBBANDS

- Coding of the subbands is done using a method and bitrate most suitable to the statistics and visual significance of that subband
- Typically 95% of the image energy is in the L frequency bands.
- Low bands may use Transform, DPCM or VQ
- Other bands may use PCM or run-length coded after coarse thresholding

Example of FIR Filter Coefficients

- Simoncelli developed the following set for orthogonal QMF filters

N	QMF-5	QMF-9	QMF-13
0	0.8593118	0.7973934	0.7737113
1	0.3535534	0.41472545	0.42995453
2	-0.0761025	-0.073386624	-0.057827797
3		-0.060944743	-0.09800052
4		0.02807382	0.039045125
5			0.021651438
6			-0.014556438

IMAGE COMPRESSION AND PACKET VIDEO

RELATIONSHIP TO WAVELET TRANSFORM CODING

- Wavelet transform coding is closely related to subband coding
  - Alternative to filter design
- Wavelet transform describes multiresolution decomposition in terms of expansion of an image onto a set of wavelet basis functions
  - Basis functions well localized in both space and time

RELATIONSHIP TO WAVELET TRANSFORM CODING

- Wavelet transform-based filters possess some regularity properties that not all QMF filters have.
  - Improved coding performance over QMF filters with same number of taps
- Subband coding lab uses Adelson and Simoncelli coefficients from SPIE paper

INTRODUCTION TO WAVELETS: HYPE

- "... the new computational paradigm [wavelets] eventually may swallow Fourier transform methods..."
- " ...a new approach to data crunching that, if successful, could spawn new computer architectures and bring about commercial realization of such difficult data-compression tasks as sending images over telephone lines."
- from "New-wave number crunching" C. Brown, Electronic Engineering Times, 11/5/90.

EARLY HISTORY OF WAVELET THEORY

- Roots found in a variety of disciplines: Mathematics, Signal Processing, Computer Vision, Physics.
- 1910 Haar basis - first wavelet.
- 1946 The Gabor transform - short time Fourier transform.
- 1964 Calderon's work on singular integral operators contains the continuous wavelet transform.
- 1971 A. Rosenfeld and M. Thurston: multi-resolution techniques invented in machine vision - multi-resolution schemes inherent in the wavelet transform.
- 1976 A. Croiser, D. Estaban, C. Galand: quadrature mirror filter banks for speech coding - digital implementation of wavelets.

RECENT HISTORY OF WAVELETS

- 1984 J. Morlet and A. Grossman ‘‘invent" wavelets apply them to the analysis of seismic signals.
- 1986 S. Mallat and Y. Meyer invent the fast wavelet algorithm.
- 1988 I. Daubechies creates a class of wavelets that can be implemented with digital FIR filters.

THE FOURIER TRANSFORM

- Analysis, Forward transform:

$$X(f) = \int x(t) \exp[-j2\pi ft] dt$$

- Synthesis, Inverse transform:

$$x(t) = \int X(f) \exp[j2\pi ft] df$$

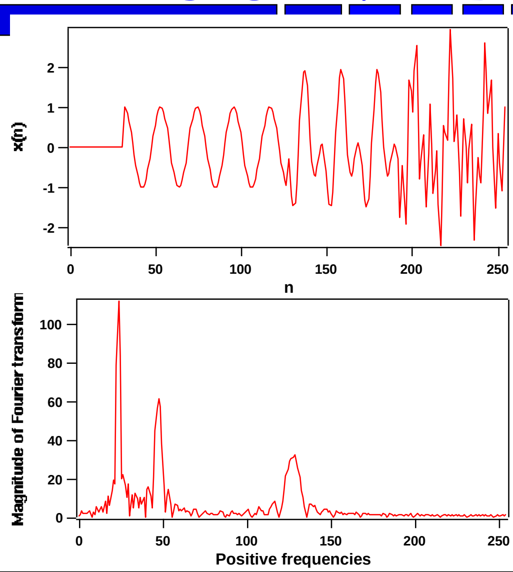
- Forward transform analyzes or decomposes x(t) into sinusoids. X(f) represents how much of the sinusoid with frequency f is in x(t).
- Inverse transform synthesizes x(t) from sinusoids, weighted by X(f).

IMAGE COMPRESSION AND PACKET VIDEO

THE FOURIER TRANSFORM PROPERTIES

- Linear Transform.
- Analysis of signals into sines and cosines has physical significance - tones, vibrations.
- Sinusoids (complex exponentials) are eigenfunctions of linear, time-invariant systems, and of differential and integral operators.
- Fast algorithms exist. The fast Fourier transform requires $O(N \log_2 N)$ computations.

PROBLEMS WITH THE FOURIER TRANSFORM: AND EXAMPLE



PROBLEMS WITH THE FOURIER TRANSFORM

- Fourier transform well-suited for stationary signals - signals that do not vary with time. This model does not fit real signals well.
- For time-varying signals or signals with abrupt transitions, the Fourier transform does not provide information on when transitions occur.
- Fourier transform has excellent frequency localization, but poor time localization.
- Need a better way to represent functions that are localized in both time and frequency.

THE SHORT-TIME FOURIER TRANSFORM

- Analysis:

$$STFT(\tau, f) = \int x(t) w^*(t-\tau) \exp[-j2\pi ft] dt$$

- Synthesis:

$$x(t) = \iint STFT(\tau, f) w(t-\tau) \exp[j2\pi ft] d\tau df$$

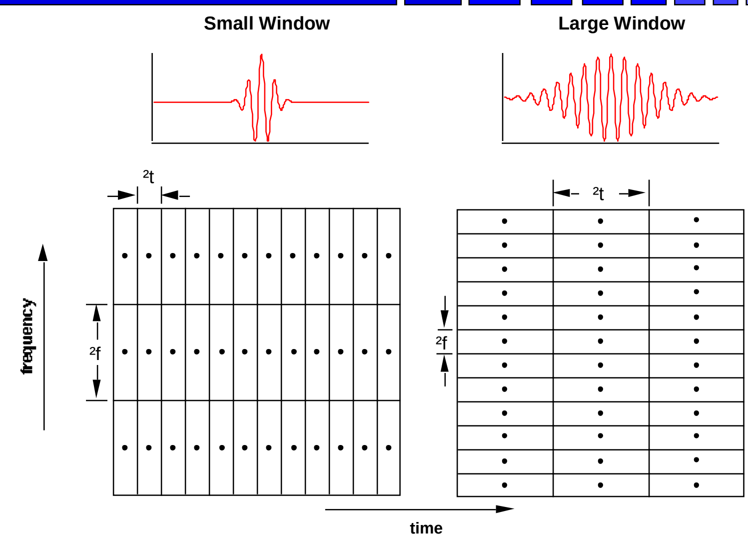
where w(t) is a window function localized in time and frequency.

- Analysis functions are sinusoids windowed by w(t).
- Common window functions: Gaussian (Gabor), Hamming, Hanning, Kaiser.

THE SHORT-TIME FOURIER TRANSFORM PROPERTIES

- Linear transform.
- Time resolution (Δt) and frequency resolution (Δf) are determined by w(t), and remain fixed regardless of τ of f.
- τ and f cannot be sampled (discretized) so that the analysis functions form an orthonormal basis.
- Biggest disadvantage: since Δt and Δf are fixed by choice of w(t), need to know a priori what w(t) will work for the desired application.

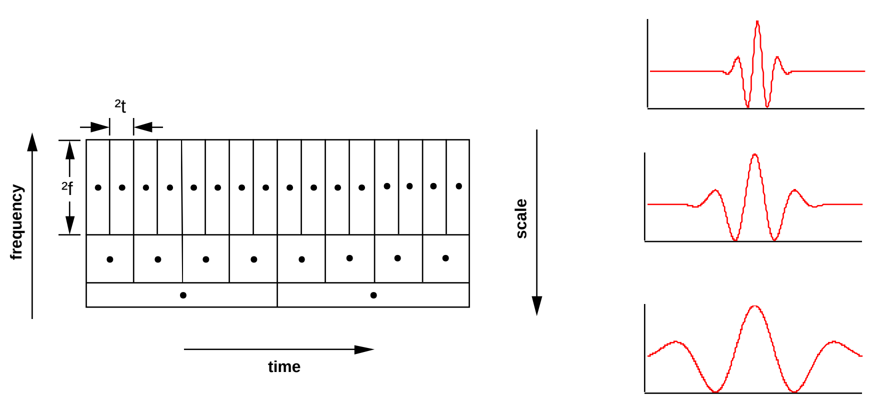
THE SHORT-TIME FOURIER TRANSFORM TIME-FREQUENCY RESOLUTION



THE WAVELET TRANSFORM TIME-FREQUENCY RESOLUTION

Fundamental Idea Behind the Wavelet Transform:
²t, ²f vary as a function of scale
(scale = 1/frequency).

Wavelet Basis functions at 3 different scales



THE WAVELET TRANSFORM: DEFINITION

- Analysis:

$$CWT(\tau, s) = \langle \Psi_{\tau,s}, x \rangle = \frac{1}{\sqrt{s}} \int x(t) \Psi\left(\frac{t-\tau}{s}\right) dt$$

- Synthesis:

$$x(t) = \frac{1}{C_h \sqrt{s}} \iint_{s>0} CWT(\tau, s) \Psi\left(\frac{t-\tau}{s}\right) dt$$

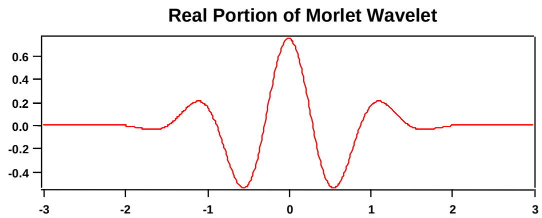
- where $C_h = \int \frac{|\Psi(f)|^2}{|f|} df < \infty$
- $\Psi(f)$ = Fourier transform of $\psi(t)$
- $\psi(t)$ is a bandpass function: $\int \Psi(t) dt = 0.$

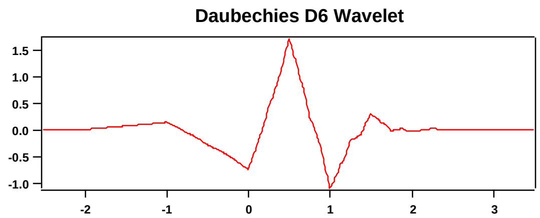
THE WAVELET TRANSFORM PROPERTIES

- Linear transform.
- All analysis functions are shifts and dilations of a single function $\psi(t)$, the wavelet.
- Time resolution and frequency resolution vary as a function of scale. Analysis functions have constant Q.
- There do exist wavelets such that s and τ can be discretized so that the $\psi_{n,k}(t)$ form an orthonormal basis.

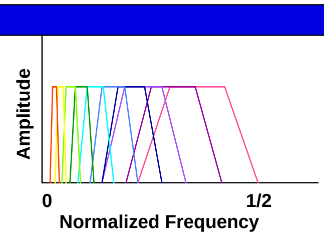
$$\Psi_{n,k}(t) = \Psi_{s_0}\left(\frac{t}{k} - n\tau_0\right)$$

EXAMPLES OF WAVELET FUNCTIONS

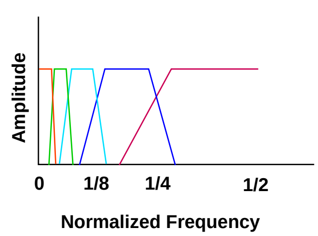




WAVELET TRANSFORM AS BANDPASS FILTERING

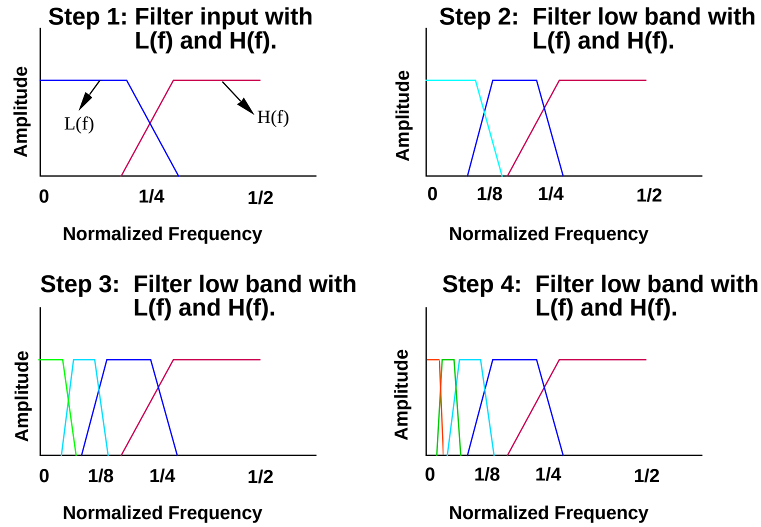


"Continuous" case
- **Many scales.**
- **Filters overlap.**
- **Redundant, but robust representation.**
- **Computationally intensive.**

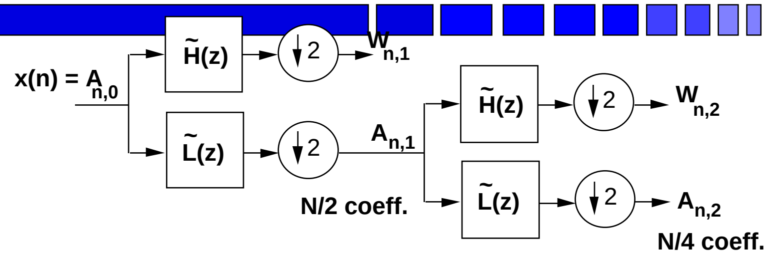


Orthogonal case
- **Dyadic scale relationship.**
- **Minimum filter overlap.**
- **Efficient signal representation.**
- **Bandpass filters = wavelets.**
- **Lowpass filter = scaling function.**
- **Used for fast algorithm.**

FAST WAVELET TRANSFORM: FREQUENCY DOMAIN REPRESENTATION



FAST WAVELET TRANSFORM FORWARD TRANSFORM



Pyramid decomposition

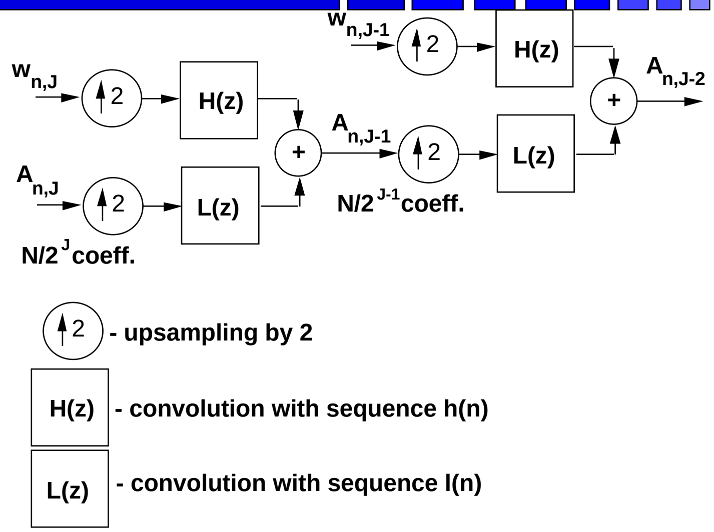
↓2 - subsampling by 2

H̃(z) - convolution with h̃(n) = h(-n)

L̃(z) - convolution with l̃(n) = l(-n)

H(z) and L(z) are known as perfect reconstruction, quadrature mirror filters.

FAST WAVELET TRANSFORM
INVERSE TRANSFORM



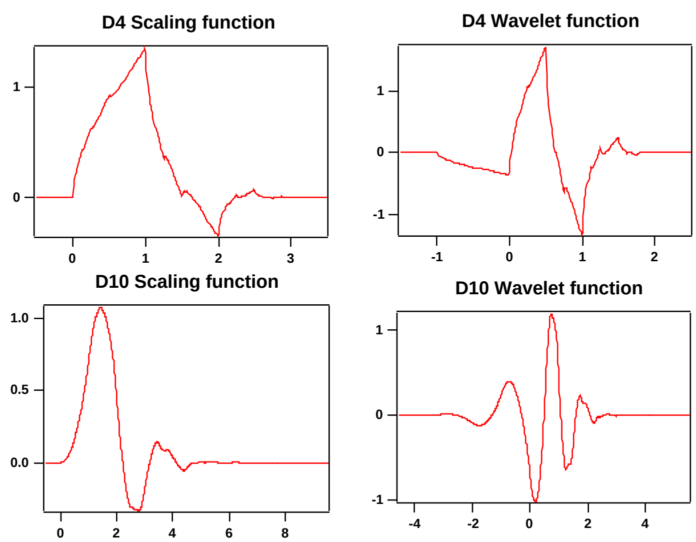
FAST WAVELET TRANSFORM PROPERTIES

- h(n) and l(n) can be purely real, so that the discrete wavelet transform is real.
- Algorithm is very fast - $O(N)$ operations.
- Discrete wavelet transform is not shift-invariant. x(n-G) will have different coefficients. A deficiency.
- Key to the algorithm is the design of h(n) and l(n). Can h(n) and l(n) be designed so that wavelets and scaling function form an orthonormal basis? Can filters of finite length be found? - YES.

DAUBECHIES FAMILY OF WAVELETS

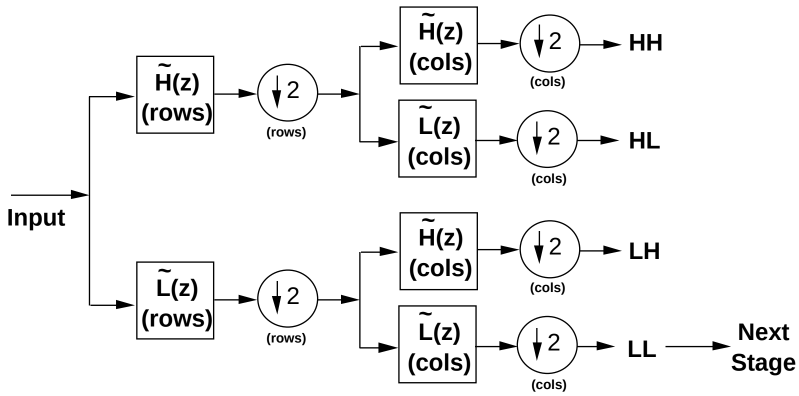
- A family of orthonormal wavelets parameterized by D (even) - the number of non-zero l(n), h(n).
- Size of wavelets: $1 - D/2 \le x \le 1 + D/2$. Size of scaling function $0 \le x \le D$.
- Smoothness of wavelet and scaling functions increase with D. The wavelet function has D/2 vanishing moments.
- Because D is finite, l(n) and h(n) can be implemented with finite impulse response digital filters.
- D = 2 is the Haar basis. D = 6 is the family of wavelets most used.

DAUBECHIES FAMILY OF WAVELETS EXAMPLES

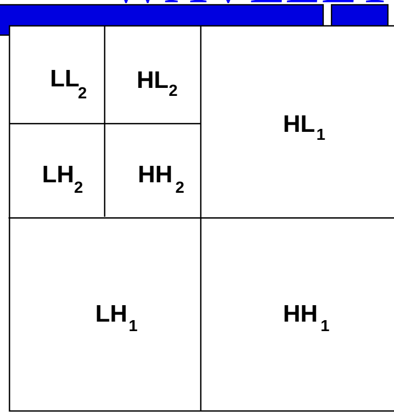


2D FAST WAVELET TRANSFORM

- **Perform 1D wavelet transform along rows, then columns.**
  - **- Separable algorithm, like FFT.**
  - **- Fast algorithm O(N), where N = rows * columns.**



COEFFICIENTS IN 2D WAVELET TRANSFORM



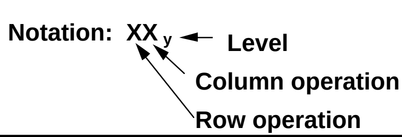
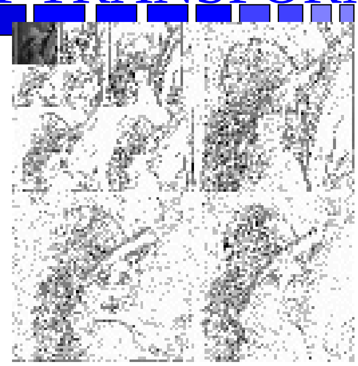
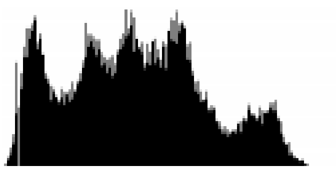
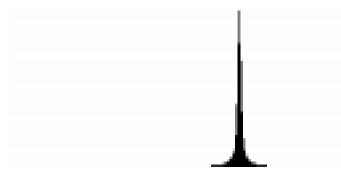


IMAGE EXAMPLE: 2D WAVELET TRANSFORM





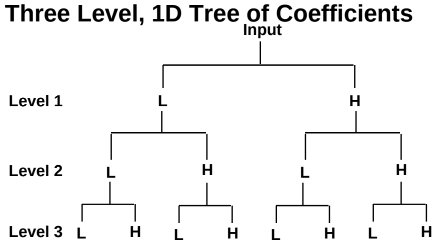


EXTENSIONS OF WAVELETS: WAVELET PACKETS

- **Approach**
  - **- Create a family (tree) of coefficients at multiple scales.**
  - **- Prune tree to create an optimized orthonormal basis.**
- **Optimzation measures**
  - **- Minimum entropy.**
  - **- Minimum number of coefficients above a threshold value.**
- **Properties**
  - **- Fast algorithm O(N log N) operations.**
  - **- Creates bases optimized for each signal: most efficient representation.**
- **Active area of research**

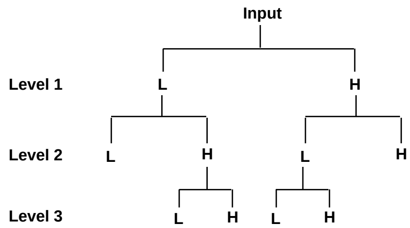
WAVELET PACKET EXAMPLE

Three Level, 1D Tree of Coefficients

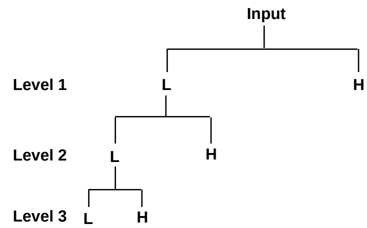


L = Lowpass filtering + subsampling.
H = Highpass filtering + subsampling.

Pruned, three level, optimum basis.



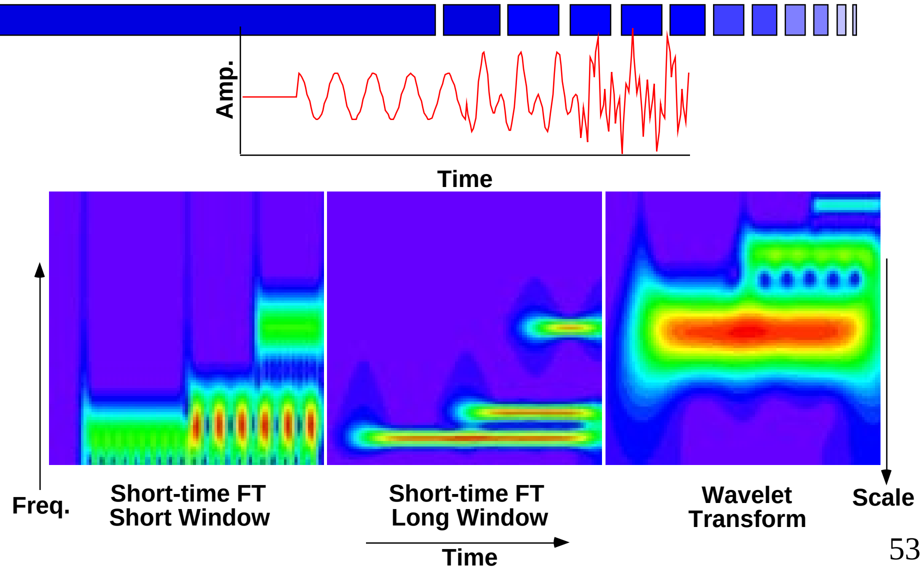
Wavelet basis



APPLICATIONS: SIGNAL ANALYSIS

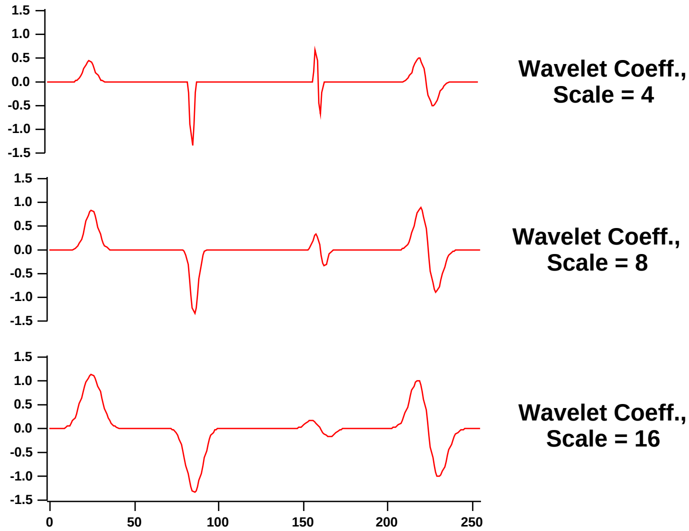
- Detection of transients
  - Use "zooming in" effect of scale change to pinpoint time of transient.
- Classification of transients or discontinuities.
  - Observe the magnitude of wavelet coeff. as a function of scale at discontinuity.
  - Applications: Classification of discontinuities in images (edges). Classification of radar scatterers. De-noising of signals and images.

SIGNAL EXAMPLE: THREE TRANSIENT SINUSOIDS



WAVELET COEFFICIENTS AT DISCONTINUITIES

WAVELET COEFFICIENTS AT DISCONTINUITIES



APPLICATIONS: MULTI-RESOLUTION PROCESSING

- Pattern Recognition
  - Determine size (scale) of important features.
  - Use same-shape properties of wavelets to examine textures, fractal signals.
  - Wavelet transform is an alternate transform that can be used in conjunction with the FFT to build feature vectors.
- Computer Vision
  - Analyze image content on a coarse-to-fine scale approach.
  - Multi-resolution approaches to edge detection, template matching, and stereo matching.

APPLICATIONS: DATA COMPRESSION

- Wavelets capture transients easily
  - Efficient representation for edges in images, transients in speech, music.
  - Edges, transients important for good visual/audio quality.
- Wavelet transform is fast so that it can be done in real time
  - Important for video.

APPLICATIONS: NUMERICAL ALGORITHMS

- Non-linear Partial Differential Equations
  - Wavelets are good basis functions for transients. Multi-grid approximation built-in.
  - Wavelet-based algorithms are more stable and converge faster.
  - Examples: shock waves, turbulence.

Wavelet Theory: Conclusions

- Wavelets are a useful tool, may be better than the Fourier transform in specific applications.
- Most powerful aspect of wavelets is their inherent multi-resolution analysis of signals and images. Wavelets have popularized multi-resolution approaches.
- Still a new field, many conferences, special journal issues, everyone is trying wavelets.
- Applications are wide-ranging. Signal processing, computer vision, data compression, quantum physics, fast numerical algorithms.

Embedded Image Coding Using Zero Trees

- Jerry Shapiro from Sarnoff Research Center (Now with Aware Inc)
- Method has been applied to Wavelets, but also can be applied to other transforms
- Method developed after recognition of the effect of quantization on transform coefficients

Benefits of Zero Tree Method

- Optimum encoding of quantized coefficients
- Efficient ROI coding
- Efficient Progressive Coding
- Ability to code to set output size
  - Use of a significance map to show all of the non-zero coefficients.

Methods for Coding Coefficients

- Hufman coding
- Vector Quantization
- Arithmetic Coding
- Organization of the coefficients is key to efficient coding
  - The low pass residual subband has the same statistics as an image
  - The other subbands are zero mean

How Do Zero Trees Work

- Transform Coefficients are losslessly encoded (after quantization)
- How we present the coefficients for coding will determine how efficient the encoding will be
- Coefficients can be ordered so that progressive transmission or region of interest is presented to the coder

Example (From Fractal Imaging (Ning Lu))

Image Sample

B =

139	144	149	153	155	155	155	155
144	151	153	156	159	156	156	156
150	155	160	163	158	156	156	156
159	161	162	160	160	159	159	159
159	160	161	162	162	155	155	155
161	161	161	161	160	157	157	157
162	162	161	163	162	157	157	157
162	162	161	161	163	158	158	158

Haar Filter Results

H(B) =

1259.6	0.1	-13.2	1.5	-6.0	-3.5	1.5	0.0
-17.4	-12.9	-0.5	5.0	-3.5	-0.5	1.5	0.0
-20.2	4.0	-3.2	0.0	-0.5	-0.5	5.0	0.0
-2.0	-3.0	1.5	0.0	0.0	-1.0	5.0	0.0
-6.0	-3.5	-2.5	-1.0	1.0	-0.5	-1.5	0.0
-7.5	0.5	-2.5	-3.0	-1.5	-2.5	0.5	0.0
-1.5	0.5	0.0	-2.0	-0.5	-0.5	2.0	0.0
0.0	1.0	1.0	-1.0	0.0	-1.0	0.0	0.0

Quantization (Threshold 9)

Q(H(B))=9*

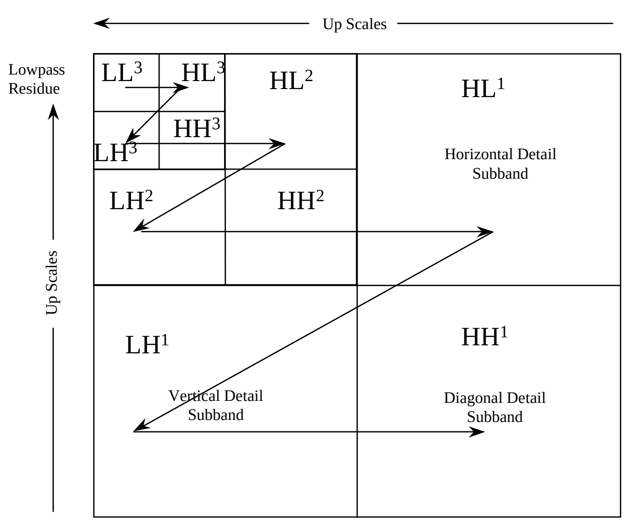
140	0	-1	0	-1	0	0	0
-2	-1	0	1	0	0	0	0
-2	0	0	0	0	0	1	0
0	0	0	0	0	0	1	0
-1	0	0	0	0	0	0	0
-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

Recovered Image

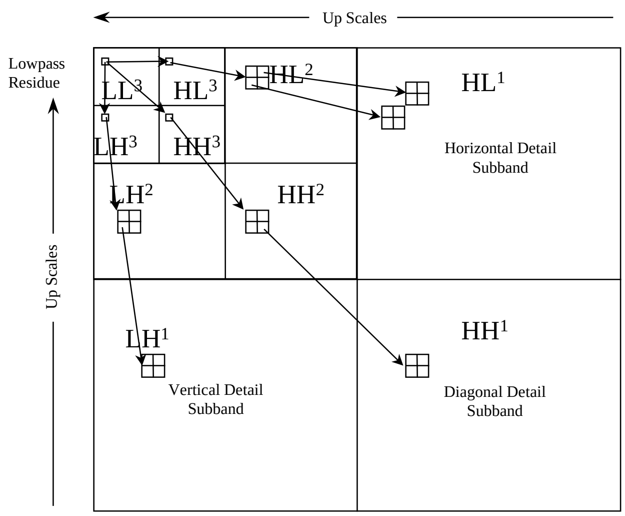
B' =

138	147	152	152	156	156	156	156
147	156	152	152	156	156	156	156
152	152	161	161	156	156	156	156
161	161	161	161	156	156	156	156
161	161	161	161	165	156	156	156
161	161	161	161	165	156	156	156
161	161	161	161	165	156	156	156
161	161	161	161	165	156	156	156

Wavelet Zero Tree

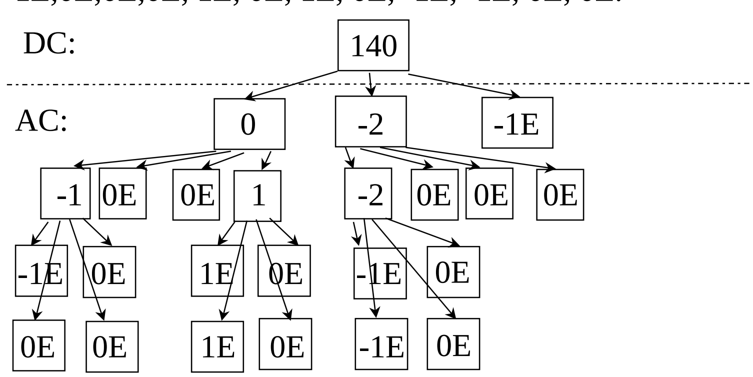


Wavelet Zero Tree



Coded Sample Images

AC Coefficient Symbol String:
0,-2,-1E; -1,0E,0E,1 -2,0E,0E,0E;
-1E,0E,0E,0E, 1E, 0E, 1E, 0E, -1E, -1E, 0E, 0E.



Features of Patented Embedded Zero Tree Coders

- Discrete wavelet transform (multiresolution)
- Zerotree coding provides a compact multiresolution representation of significance maps
- Successive Approximation of coefficients by combining sub-bands provides a compact multiprecision representation

Successive-Approximation Quantization

- Coding of multiple significance maps using zero trees
- Coding or decoding can stop at any point in the process.
  - Coding for fixed bit rate output
  - Coding for progressive output.
- SAQ sequentially applies sequence of thresholds to determine significance.

SAQ Details

- During encoding an decoding, two separate lists of wavelet coefficients are maintained
- At any point in the process, a dominate list contains the coordinates of those coefficients that have not yet been found to be significant.
  - The table is ordered such as the subbands are ordered, and each set of coefficients are ordered.

SAQ Details

- All coefficients on each subband list appear on the initial dominant list prior to coefficients on the next subband.
- The subordinate list contains the magnitudes of the coefficients that are found to be significant.
- For each threshold, each list is scanned once.
- During each dominate pass, the coefficients are compared to the threshold value to see if they are significant.

Features of Patented Embedded Zero Tree Coders

- Prioritization protocol whereby the ordering of importance is determined, in order by precision, magnitude, scale, and spatial location of the wavelet coefficients.
  - Arithmetic Coder of coefficients
- Algorithm runs sequentially and stops whenever target bit rate or target distortion is met.
  - Significance map is followed to build coefficients.

How Do We Code To Fixed Bit Rate

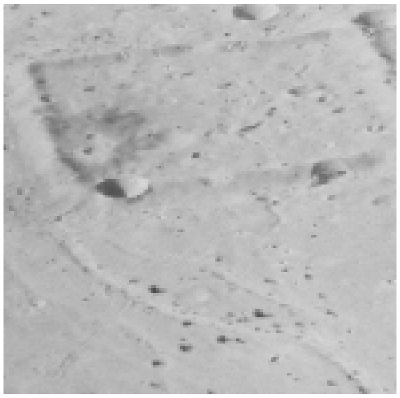
- VQ of coefficients can operate at fixed bit rate
- Dynamic Adjustment of quantized coefficients toward target bit rate
- Dynamic adjustment of significance map to reach target bit rate.
  - Relies on efficient tree searching methods and SAQ methods

Wavelet Compression Results

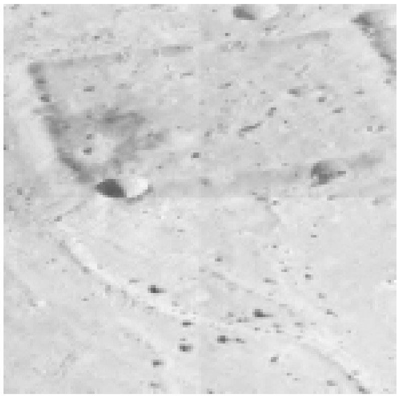
(a) Desert Overview Image - Original 512x512

(b) Desert Overview Image at 13.5:1 Compression Ratio

Wavelet Compression Results



(a) Desert Closeup Image - 512x512



(b) Desert Closeup Image at 32.25:1 Compression Ratio

Wavelet Compression Results

(a) Runway Overview Image - 512x512

(b) Runway Overview Image - 14.5:1 Compression Ratio

IMAGE COMPRESSION AND PACKET VIDEO

Wavelet Compression Results

(a) Runway Closeup Image 512x512

(b) Runway Closeup Image at 19.75:1 Compresssion Ratio